HEAT TRANSFER IN LAMINAR FLOW. II.*
THREE-PARAMETER DESCRIPTION FOR DUCTS OF SIMPLE CROSS-SECTION

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On the example of heat transfer into the non-Newtonian liquid flowing through a tube, annular and flat ducts is demonstrated that for calculation of dimensionless mean mixing temperature and of the Nusselt numbers there exist pairs of limiting relations valid exactly for exchangers of zero respectively infinite lengths. The approximative integral description can be based on these relations, which is illustrated here in a graphical form.

Though a number of solutions has been already published of equation for laminar heat transfer in various geometries and types of liquids and enough experimental material has been accumulated[1] no general equation exists for describing the heat transfer in the whole exchanger. Relations with a number of constants which are the result of mathematical analysis of the problem, are for common use rather clumsy and furthermore they are not adequate in the region of the inlet section. Simple and useful relations exist for both extreme cases, for very long and very short exchangers. However, until now the limits of validity of these relations have not been determined so that they are not usable without risk for judging the process in exchangers of medium length. Here it is shown that in the whole class of cases concerning the heat transfer into non-Newtonian liquid in annular ducts[2] the both limiting relations can be combined for a simple description of the whole extent of heat transfer[3].

DESCRIPTION OF HEAT TRANSFER IN ANNULAR DUCTS

Let us consider the section of annular duct into which enters the liquid at temperature T_0. In this section is the wall of radius R kept at the temperature $T_w \neq T_0$ and the wall of radius $\varkappa R$ is insulated. If the axial distance is x then the liquid flowing through the considered area $x = \text{const.}$ should have after adiabatic mixing the temperature $T_M(x)$, which is called the mean mixing temperature, given by

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$$T_M(x) = (1/Q) \int_{\varkappa R}^{R} 2\pi r\, v(r)\, T(r, x)\, dr\,. \tag{1}$$

The dimensionless mean mixing temperature

$$t_M(z) = (T_M(x) - T_w)/(T_0 - T_w) \tag{2}$$

is the quantity giving the ratio of the local temperature difference to the original temperature difference between the wall and the liquid. The heat transferred through the wall per unit of time between the origin and the section x = const. equals to

$$q(x) = (T_M(x) - T_0)\, \varrho c_p Q\,. \tag{3}$$

If we relate this quantity also to the area of heat transfer wall $2\pi Rx$ and to the characteristic temperature difference ΔT, we get the mean heat transfer coefficient. By its multiplication by the characteristic transverse dimension chosen as $2R|1 - \varkappa|$ (so that it has for the tube the usual value) and by division by the coefficient of thermal conductivity k we obtain the mean Nusselt number. Certain subjectivity lies in the form of introducing the characteristic temperature difference ΔT, where usually the arithmetic mean of temperature differences between the liquid and the wall at the origin and the point x is used, so that

$$\Delta T_A = [(T_w - T_0) + (T_w - T_M(x))]/2\,, \tag{4}$$

which leads to the common definition of arithmetic mean Nusselt number

$$\mathrm{Nu}_{AM} \equiv \frac{q\,.\,2|1 - \varkappa|}{\pi k x(2T_w - T_0 - T_M(x))} \tag{5}$$

or to the logarithmic mean

$$\Delta T_L = \frac{(T_w - T_M(x)) - (T_w - T_0)}{\ln\,[(T_w - T_M(x))/(T_w - T_0)]}\,, \tag{6}$$

from which follows the definition of less frequently used logarithmic mean Nusselt number

$$\mathrm{Nu}_{LM} \equiv \frac{q|1 - \varkappa|\,\ln\,[(T_w - T_M(x))/(T_w - T_0)]}{\pi k x(T_0 - T_M(x))}\,. \tag{7}$$

After introducing the dimensionless axial coordinate for the annular duct in the same way as in the first part of this series[2] $z = x\,.\,k/[\varrho\,.\,c_p\,.\,U\,.\,R^2\,.\,(1 - \varkappa)^2]$, we can write equations relating mutually all the introduced integral characteristics. Thus

$$\mathrm{Nu}_{\mathrm{AM}} = [(1 - t_{\mathrm{M}})/(1 + t_{\mathrm{M}})]\,[2(1 + \varkappa)/z] \tag{8}$$

and

$$\mathrm{Nu}_{\mathrm{LM}} = [(1 + \varkappa)/z] \ln t_{\mathrm{M}}\,. \tag{9}$$

It is advisable to use the Nusselt number in cases where the heat transfer through the wall is actually studied, for example in experimental arrangements in which the enthalpy balance is made on the tube side and further where the heat transfer coefficients on both sides of the wall are compared. The mean mixing temperature is not usually measured directly because difficulties of experimental character are usually encountered (especially with highly viscous liquids). However, the mean temperature is commonly used in mathematical solution of the problem. The advantage of dimensionless mean mixing temperature as an integral characteristics is that apart from providing quantitative data (enabling balancing) it provides also qualitative information as to what extent the temperature of the liquid approaches the temperature of the wall and how intensively the heat is still transferred.

Shape of Integral Characteristics

In the last paper of this series[2] has been derived, that the relation

$$t_{\mathrm{M}}(z) = \sum_{i=1}^{\infty} t_{\mathrm{M}i} \exp\left[-b_i^2 z\right] \tag{10}$$

exactly holds, where $t_{\mathrm{M}i}$ and b_i^2 are constants of solution of the differential heat

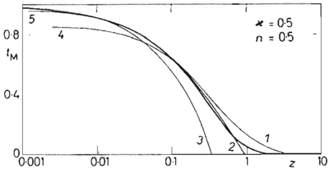

Fig. 1

Dimensionless Mean Mixing Temperature t_{M} as a Function of Dimensionless Length z for Heat Transfer into the Power-Law Liquid $n = 0.5$ in Annular Duct $\varkappa = 0.5$

Solid line represents the exact solution; *1* Eq. (*15*) extrapolated for $\mathrm{Nu}_{\mathrm{LM}}$ Eq. (*16*), *2* Eq. (*15*) extrapolated for $\mathrm{Nu}_{\mathrm{AM}}$, *3* extrapolated Eq. (*12*), *4* first term of series (*11*) (Eq. (18)), *5* series (*11*) with $N = 3$.

transfer equation. For a chemical engineer it is usually sufficient to obtain the required information with a $\pm 5\%$ accuracy. In the case of dimensionless mean mixing temperature we have, in order to be able to make with sufficient accuracy also the enthalpy balances (for example with the use of Eq. (*8*)), to take for 100% smaller values of t_M and $(1 - t_M)$. For attaining such accuracy we do not have to carry out addition of the whole infinite series (*10*) but the finite series

$$t_M(z) \approx \sum_{i=1}^{N} t_{Mi} \exp\left[-b_i^2 z\right], \tag{11}$$

suffices, where, as we have found by the study of several cases, is sufficient to substitute for N the integer of the value closest to the expression $0{\cdot}3/\sqrt{z}$. From these considerations it is obvious that for description of the heat transfer in the inlet section of the heat exchanger or in cases when the exchanger is short, even the approximate calculation is subject to knowledge of a great number of constants. Without regard to difficulties related with determination of these constants and their tabelation, evaluation of the series (*11*) would be very laborious. Therefore we are looking for a more suitable approximative expression of the dependence $t_M(z)$ for small values of z.

Let us begin with the assumption, that the temperature differences in the inlet section affect practically only a thin layer adjacent to the wall. Under such circumstances can be applied the idea of Léveque[4] according to which for determination of the temperature profile in this thin layer has no importance what takes place behind its boundary. Solution of the temperature profile is then similar to the solution of the temperature profile for laminar flow around a flat plane. By application of this analogy to the studied case of annular duct we obtain the expression[5]

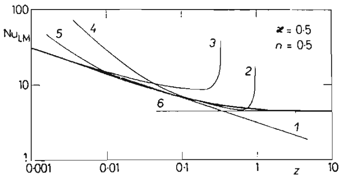

FIG. 2

Logarithmic Mean Nusselt Number for $\varkappa = 0{\cdot}5$ and $n = 0{\cdot}5$

Curves are denoted as those in Fig. 1; *6* $Nu_{LM} = (1 + \varkappa)\, b_1^2$.

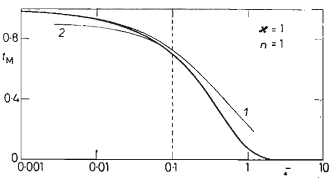

FIG. 3

Dependence of t_M on z for $\varkappa = 1$ and $n = 1$

Solid line represents the exact solution; *1* Eq. (*17*), *2* Eq. (*18*).

$$\lim_{z \to 0} t_M = 1 - \frac{h(w_0')^{1/3}}{1 + \varkappa} z^{2/3}, \tag{12}$$

where h is the numerical constant given by

$$h \equiv 3^{1/3}/\Gamma(^4/_3) = 1{\cdot}61510\ldots \tag{13}$$

and w_0' is the dimensionless velocity gradient on the heat transfer wall

$$w_0' = \left| \frac{R(1 - \varkappa)}{u} \left(\frac{dv}{dr} \right)_{r=R} \right|. \tag{14}$$

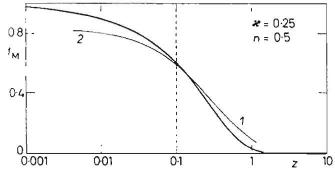

FIG. 4

Dependence of t_M on z for $\varkappa = 0{\cdot}25$ and $n = 0{\cdot}5$

Curves are denoted as those in Fig. 3.

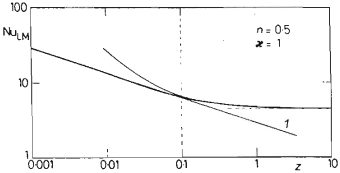

FIG. 5

Dependence of Nu_{LM} on z for $\varkappa = 0{\cdot}5$ and $n = 1$

Solid line represents the exact solution; *1* Eq. (*16*), 2 Eq. (*19*).

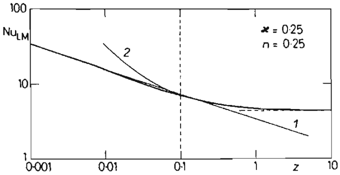

FIG. 6

Dependence of Nu_{LM} on z for $\varkappa = 0{\cdot}25$ and $n = 0{\cdot}25$

Curves are denoted as those in Fig. 5.

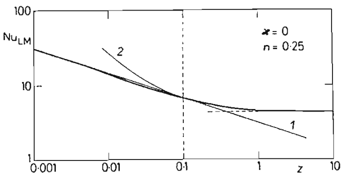

FIG. 7

Dependence of Nu_{LM} on z for $\varkappa = 0{\cdot}5$ and $n = 0{\cdot}25$

Curves are denoted as those in Fig. 5.

For the Nusselt numbers holds

$$\lim_{z\to 0} \mathrm{Nu}_{AM} = \lim_{z\to 0} \mathrm{Nu}_{LM} = h(w_0')^{1/3}\, z^{-1/3} . \qquad (15)$$

The given limiting relations (*12*), (*15*) excellently express the heat transfer in the heat exchanger of length $z < 0{\cdot}01$; the deviations from the accurate solution are well below the limit of 1%. For higher z, these relations tend to differ slightly and it is necessary to decide which of the relation is to be used for approximation of the function $t_M(z)$. From comparison of a number of examples and due to the fact that the extrapolated relation for logarithmic mean Nusselt number is the only one according to which counts with the course of the function $t_M(z)$ are within the limits 1 to 0 (*i.e.* in agreement with the actual situation), we prefer the approximative expression

$$\mathrm{Nu}_{LM}(z) \approx h(w_0')^{1/3} z^{-1/3} \quad \text{for} \quad z \leqq 0{\cdot}1 . \qquad (16)$$

In terms of the mean mixing temperature this means

$$t_w(z) \approx \exp\left[-\frac{h(w_0')^{1/3}}{1+\varkappa}\, z^{2/3}\right] \quad \text{for} \quad z \leqq 0{\cdot}1 . \qquad (17)$$

For higher values of z from expression (*11*) follows the validity of the approximate relation

$$t_M(z) \approx t_{M1} \, . \exp\left[-b_1^2 z\right] \quad \text{for} \quad z \geqq 0{\cdot}1 , \qquad (18)$$

or

$$\mathrm{Nu}_{LM}(z) \approx \frac{1+\varkappa}{z} \ln t_{M1} + (1+\varkappa)\, b_1^2 \quad \text{for} \quad z \geqq 0{\cdot}1 . \qquad (19)$$

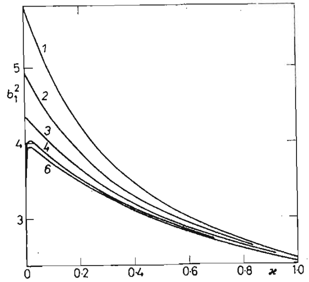

FIG. 8

Dependence of the First Eigenvalue of the Given Problem b_1^2 on $\varkappa$ and n

1 $n = 0$ (plug flow), *2* $n = 0{\cdot}1$, *3* $n = 0{\cdot}25$, *4* $n = 0{\cdot}5$, *5* $n = 0{\cdot}75$, *6* $n = 1$ (Newtonian flow).

From the example given in Figs 1 and 2 can be seen how individual approximations are close to the exact solution. In Figs 3 to 7 is the exact solution compared with approximative formulas (*16*), (*19*) eventually (*17*), (*18*); it is obvious that the sign of inexactness in these relations does not mean considerable errors in the results. In conclusion we can say that for integral description of heat transfer in flow of the non-Newtonian power-law fluid in a general annular duct (as limiting situations of the flat duct and pipe inclusive) with the temperature step change on the outside wall, only three constants w'_0, b_1^2 and t_{M1} are sufficient, whose dependence on the geometrical simplex of the annulus $\varkappa$ and of the flow index n is given in Figs 8–10. In using this integral description in no case an error exceeding 5% is made.

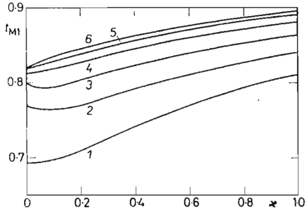

FIG. 9

Dependence of the First Constant of Series (*10*) t_{M1} on $\varkappa$ and n

Curves are denoted as those in Fig. 8.

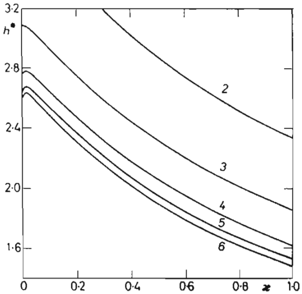

FIG. 10

Dependence of Expression $h^* = h(w'_0)^{1/3} : (1 + \varkappa)$ on $\varkappa$ and n

Curves are denoted as those in Fig. 8.

LIST OF SYMBOLS

c_p	specific heat (cal g^{-1} deg^{-1})
k	thermal conductivity of liquid (cal cm^{-1} s^{-1} deg^{-1})
Q	volumetric flow rate of liquid (cm^3 s^{-1})
q	heat flux (cal s^{-1})
r	radial coordinate (cm)
R	radius of heat transfer wall (cm)
T	temperature (°C)
T_M	mean mixing temperature (°C)
T_0	temperature of inlet liquid (°C)
T_w	wall temperature (°C)
ΔT_L	logarithmic mean of temperature differences (°C)

ΔT_A arithmetic mean of temperature differences (°C)
v liquid velocity (cm s^{-1})
x axial distance from the point of temperature step change (cm)
ϱ liquid density (g cm^{-3})

Dimensionless quantities

b_i^2 eigenvalues
h numerical coefficient (*13*)
h^* multiple coefficient in Eq. (*12*)
n flow index of the power-law liquid
N number of terms of the series (*11*)
Nu_{LM} logarithmic mean Nusselt number (*7*)
Nu_{MA} arithmetic mean Nusselt number (*5*)
t_M mean mixing temperature (*2*)
t_{Mi} expansion coefficient (*10*)
w'_0 velocity gradient on the heat transfer wall (*14*)
z axial coordinate
$\varkappa$ parameter of annulus
Γ gamma function

REFERENCES

1. Knudsen J. G., Katz D. L.: *Fluid Dynamics and Heat Transfer*. McGraw-Hill, New York 1958.
2. Wichterle K., Wein O., Ulbrecht J.: This Journal, *37* 1816 (1972).
3. Wein O., Wichterle K.: Paper presented at the 3rd International CHISA Congress, Mariánské Lázně, September 1969.
4. Léveque M.: Ann. Mines *13*, 201 (1928).
5. Wichterle K.: *Thesis*. Czechoslovak Academy of Sciences, Prague 1968.

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